LIMIT DISTRIBUTIONS OF VERTEX DEGREES IN A CONDITIONAL CONFIGURATION GRAPH

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The configuration graph where vertex degrees are independent identically distributed random variables is often used for modeling of complex networks such as the Internet. We consider a random graph consisting of $N$ vertices. The random variables $\eta_1, \ldots, \eta_N$ are equal to the degrees of vertices with the numbers $1, \ldots, N$. The probability $P\{\eta_i = k\}, i = 1, \ldots, N, k = 1, 2, \ldots$, is equivalent to $h(k)/k^\tau$ as $k \to \infty$, where $h(x)$ is a slowly varying function integrable in any finite interval, $\tau > 1$. We obtain the limit distributions of the maximum vertex degree and the number of vertices with a given degree under the condition that the sum of degrees is equal to $n$ and $N, n \to \infty$.

Key words: configuration graph; the limit distribution; vertex degree.

INTRODUCTION

The study of random graphs has been causing growing interest in connection with the wide use of these models for the description of complex networks (see, e. g. [3, 6, 11]). Such models can be used to adequately describe the topology of transport, electricity, social, telecommunication networks and global Internet. Observations on real networks showed that their topology can be described by random graphs with vertex degrees being independent identically distributed random variables with power-law distribution. In [3] it was suggested that for large $k$ the number of vertices with the degree $k$ is proportional to $k^{-\tau}$, where $\tau > 1$. That is why in [11] it was suggested that the distribution of the vertex degree $\eta$ is...
\( \mathbf{P}\{\eta \geq k\} = h(k)k^{-\tau + 1}, \quad k = 1, 2, \ldots, \) (1)

where \( h(k) \) is a slowly varying function.

We consider a random graph consisting of \( N+1 \) vertices. Let random variables \( \eta_1, \ldots, \eta_N \) be equal to the degrees of vertices with the numbers \( 1, \ldots, N \). Each vertex is assigned a certain degree in accordance with the distribution (1). The vertex degree is the number of stubs (or semiedges) that are numbered in an arbitrary order. Stubs are vertex edges for which adjacent nodes are not yet determined. The vertex 0 is auxiliary and has degree 0 if the sum of all other vertices is even, else the degree is 1. It is clear that we need to use the auxiliary vertex 0 for the sum of degrees to be even. The graph is constructed by joining all the stubs pairwise equiprobably to form links.

There are many papers where the results describing the limit behaviour of different random graph characteristics were obtained. The authors of [11] were sure (without proof) that the function \( h(k) \) in (1) does not influence the limit results, and that to study the configuration graph one can replace \( h(k) \) with the constant 1. In our work we will show that the role of the slowly varying function \( h(k) \) is more complicated.

We consider the subset of random graphs under the condition \( \eta_1 + \cdots + \eta_N = n \). Such conditional graphs can be useful for modeling of networks for which we can estimate the number of communications. They are useful also for studying networks without conditions on the number of links by averaging the results of conditional graphs with respect to the distribution of the sum of degrees. Conditional random graphs were first analyzed in [9], where \( h(k) \equiv 1 \). Obviously, the limit behaviour of a random graph depends on the degree structure. In [9] the limit distributions were obtained for the maximum vertex degree and the number of vertices of a given degree as \( N \) and \( n \) tend to infinity in such a way that \( 1 < n/N < \zeta(\tau) \), where \( \zeta(\tau) \) is the value of the Rimman’s zeta-function at the point \( \tau \). For other zones of parameters analogous results were obtained in papers [7, 8, 10].

Here we extend the results on the maximum vertex degree and the number of vertices of a given degree to the configuration graphs with degree distribution (1), where \( h(k) \) is not constant. In the following section the main results are formulated, then auxiliary statements are proved. And the last section contains proofs of the main results.

**Main Results**

In the paper we assume that the distributions of node degrees are

\[ p_k = \mathbf{P}\{\eta_i = k\} = \frac{h(k)}{k^\tau\Sigma(1, \tau)}, \quad (2) \]

where \( i = 1, \ldots, N, \quad k = 1, 2, \ldots, \quad \tau > 1, \quad h(k) \) is a slowly varying function integrable in any finite interval and

\[ \Sigma(x, y) = \sum_{k=1}^{\infty} x^k h(k) k^{-y}. \quad (3) \]

We denote also by \( \xi_1, \ldots, \xi_N \) the auxiliary independent identically distributed random variables such that

\[ p_r(\lambda) = \mathbf{P}\{\xi_i = k\} = \lambda^k p_k(1, \tau)/\Sigma(\lambda, \tau), \quad (4) \]

where \( i = 1, \ldots, N, \quad k = 1, 2, \ldots \) and the parameter \( \lambda = \lambda(n, N) \) belongs to the interval \( (0, 1) \). From (2)–(4) we obtain

\[ m = \mathbf{E}\xi_1 = \Sigma(\lambda, \tau - 1)/\Sigma(\lambda, \tau), \]

\[ \sigma^2 = \mathbf{D}\xi_1 = \Sigma(\lambda, \tau - 2)/\Sigma(\lambda, \tau) - m^2. \quad (5) \]

Let the parameter \( \lambda = \lambda(n, N) \) of the distribution (4) be determined by the relation

\[ m = \Sigma(\lambda, \tau - 1)/\Sigma(\lambda, \tau) = n/N. \quad (6) \]

We denote by \( \eta(N) \) and \( \mu_r \) the maximum vertex degree and the number of vertices with the degree \( r \), respectively. We get the following results.

**Theorem 1.** Let \( n, N \to \infty, n/N \to 1, (n - N)^3/N^2 \to \infty \) and let \( r \) be such that

\[ \frac{N\lambda^{r-1}h(r)}{r^\tau} \to \infty, \quad \frac{N\lambda^{r+1}h(r + 1)}{\Sigma(\lambda, (r + 1)^\tau)} \to \gamma, \]

where \( \gamma \) is a nonnegative constant. Then

\[ \mathbf{P}\{\eta(N) = r\} \to e^{-\gamma}, \]

\[ \mathbf{P}\{\eta(N) = r + 1\} \to 1 - e^{-\gamma}. \]

We introduce the conditions:
such a way that

\[
\frac{N\lambda^{r+1}h(r+1)}{(r+1)^{\tau+1}N^{\tau}(1-\lambda)} \to \gamma,
\]

where \(\gamma\) is a positive constant. Then

\[
P\{\eta_N \leq r\} = e^{-\gamma}(1+o(1)).
\]

**Theorem 2.** Let \(N, n \to \infty, n/N \not\to \Sigma(1, \tau - 1)/\Sigma(1, \tau)\), one of the following conditions (A1) – (A5) is fulfilled, and \(r = r(N, n)\) takes values in such a way that

1. \(n/N \to 1, \quad r = 1, 2, \quad (n-N)^2/N \to \infty\);
2. \(n/N \to 1, \quad r \geq 3, \quad N\lambda^{r-1} \to \infty\);
3. \(n/N \not\to \Sigma(1, \tau - 1)/\Sigma(1, \tau)\), parameters \(\tau, N, n\) are determined by one of the conditions (A1) – (A5).

Then for a nonnegative integer \(k\) uniformly with respect to \(u = (k - Np_r(\lambda))/\sqrt{\sigma_{rrN}}\) lies in any fixed finite interval

\[
P\{\mu_r = k\} = \frac{1}{\sigma_{rrN}} e^{-u^2/2}(1+o(1)),
\]

where

\[
\sigma_{rr}^2 = p_r(\lambda) \left(1 - p_r(\lambda) - \frac{(n/N - r)^2}{\sigma^2}p_r(\lambda)\right).
\]

**Theorem 4.** Let \(n, N \to \infty, n/N \to 1, n - N \to \infty, r \geq 2\). Then

\[
P\{\mu_r = k\} = \frac{1+o(1)}{k!} (Np_r(\lambda))^k \exp \{-Np_r(\lambda)\}
\]

uniformly with respect to \((k - Np_r(\lambda))/\sqrt{\sigma_{rrN}}\) lies in any fixed finite interval. This assertion remains true for \(r \to \infty\) if \(1 < n/N < \Sigma(1, \tau - 1)/\Sigma(1, \tau)\) under one of the following conditions:

1. \(n/N \to 1, \quad n - N \to \infty\);
2. \(n/N \not\to \Sigma(1, \tau - 1)/\Sigma(1, \tau)\), parameters \(\tau, N, n\) are determined by one of the conditions (A1) – (A5).

**Remark.** In [2], a case of these theorems under the condition \(1 < C_1 \leq n/N < C_2 < \Sigma(1, \tau - 1)/\Sigma(1, \tau)\) was proved.

**Auxiliary Results**

We prove some auxiliary statements (Lemmas 1–6), and use them to prove Theorems 1–5. The technique of obtaining these theorems is based on the generalized allocation scheme suggested by V. F. Kolchin [5]. It is readily seen that for our subset of random graphs

\[
P\{\eta = k_1, \ldots, \eta = k_N\} = P\{\xi_1 = k_1, \ldots, \xi_N = k_N|\xi_1 + \ldots + \xi_N = n\}.
\]

Therefore, the conditions of the generalized allocation scheme are valid (see [5]). Let \(\xi_1^{(r)}, \ldots, \xi_N^{(r)}\) and \(\tilde{\xi}_1^{(r)}, \ldots, \tilde{\xi}_N^{(r)}\) be two sets of independent identically distributed random variables such that

\[
P\{\xi_1^{(r)} = k\} = P\{\xi_1 = k|\xi_1 \leq r\},
\]

(7)

\[
P\{\tilde{\xi}_1^{(r)} = k\} = P\{\xi_1 = k|\xi_1 \neq r\}, \quad k = 1, 2, \ldots
\]

We also put

\[
\tilde{\xi}_N = \xi_1 + \ldots + \xi_N, \quad \xi^{(r)}_N = \xi_1^{(r)} + \ldots + \xi_N^{(r)}, \quad \tilde{\xi}_N^{(r)} = \tilde{\xi}_1^{(r)} + \ldots + \tilde{\xi}_N^{(r)}.
\]

It is shown in [5] that

\[
P\{\eta_N \leq r\} = (1 - P_r)^N P\{\xi_N^{(r)} = n\} P\{\xi_N = n\}.
\]

(8)

\[
P\{\mu_r = k\} = \binom{N}{k} p^k_r(\lambda)(1 - p_r(\lambda))^{N-k} \times
\]

\[
P\{\tilde{\xi}_N^{(r)} = n - kr\} \times P\{\xi_N = n\}.
\]

(9)

From (2)–(6) we can deduce the next lemma.

**Lemma 1.** Let \(n, N \to \infty\). The next assertions are true:
1. if $n/N \to 1$ then $\lambda = ((n/N - 1)p_1/p_2)(1 + o(1))$;

2. if $n/N \not\to \Sigma(1, \tau - 1)/\Sigma(1, \tau)$ then $\lambda \to 1$.

Let us consider the limit behaviour of $\zeta_N$.

**Lemma 2.** Under the conditions of Theorems 1–4

\[ P\{\zeta_N = k\} = 1 + o(1) \exp \left\{ -\frac{(k - n)^2}{2\sigma^2 N} \right\} \]

uniformly with respect to integers $k$ such that $(k - n)/(\sigma\sqrt{N})$ lies in any fixed finite interval.

**Proof.** Let $\varphi(t)$ be the characteristic function of the random variable $\xi_1$. Then

\[ \varphi(t) = \Sigma(e^{it\lambda}, \tau)/\Sigma(\lambda, \tau). \quad (10) \]

Further we will need an explicit form of the third derivative of $\ln \varphi(t)$. From (4) it is not hard to get that

\[ \frac{\partial^3 \varphi(t)}{\partial t^3} = \left( -\frac{\Sigma(e^{it\lambda}, \tau - 3)}{\Sigma(e^{it\lambda}, \tau)} + \frac{3\Sigma(e^{it\lambda}, \tau - 2)\Sigma(e^{it\lambda}, \tau - 1)}{\Sigma^2(e^{it\lambda}, \tau)} - \frac{2\Sigma^3(e^{it\lambda}, \tau - 1)}{\Sigma^3(e^{it\lambda}, \tau)} \right). \quad (11) \]

Let $n/N \to 1$. From (2)–(4) it is easy to obtain that

\[ \sigma^2 = O(\lambda), \quad \|\ln \varphi(t)\|_m \leq C_3\lambda. \quad (12) \]

Let $\varphi_N(t)$ be the characteristic function of the random variable $(\zeta_N - n)/(\sigma\sqrt{N})$. Then

\[ \ln \varphi_N(t) = -\frac{\text{int}}{\sigma\sqrt{N}} + N \ln \varphi\left( \frac{t}{\sigma\sqrt{N}} \right) = -\frac{t^2}{2} + \frac{t^3Q(t/(\sigma\sqrt{N}))}{6\sigma^3\sqrt{N}}. \quad (13) \]

Then from Lemma 1, (12) and (13) follows relation

\[ \ln \varphi_N(t) = -t^2/2 + o(1). \quad (14) \]

Let $n/N \not\to \Sigma(1, \tau - 1)/\Sigma(1, \tau)$. It is well known (see e.g. [4]) that the slowly varying function integrable in any finite interval has the following properties:

1. $h(x) > 1/\sqrt{x}$ for sufficiently large $x$;
2. $\lim_{x \to \infty} h(x + t)/h(x) = 1, \quad t > 0$;
3. $\lim_{x \to \infty} h(x)/x^\epsilon = 0, \lim_{x \to \infty} h(x)x^\epsilon = \infty$ for any $\epsilon > 0$;
4. $h(x) = c(x) \exp\left\{ \int_{x}^{\infty} \varphi(\epsilon(t))/tdt \right\}$, where $\alpha > 0, c(x) \to c \neq 0, \varphi(x) \to 0$, as $x \to \infty$.

Using the properties (15) and Lemma 1 we can deduce that for $j = 0, 1, 2, 3$

\[ |\Sigma(e^{it\lambda}, \tau - j)| \leq C_4\lambda\Phi(\lambda, \tau - j, 1) + \lambda\Phi(\lambda, \tau - j, \epsilon, 1), \quad (16) \]

\[ |\Sigma(e^{it\lambda}, \tau)| \geq C_5 \quad \text{as} \quad t \to 0, \quad (17) \]

where $\Phi(x, s, a)$ is the Lerch transcendent function:

\[ \Phi(x, s, a) = \sum_{k=0}^{\infty} x^k(k + a)^{-s}. \quad (18) \]

It is well known (see e.g. [1]) that for the Lerch transcendent function the following properties are valid:

1. $\lambda\Phi(\lambda, 1, 1) = -\ln(1 - \lambda)$;
2. $(1 - \lambda)\Phi(\lambda, 0, 1) = 1$;
3. $(1 - \lambda)\Phi(\lambda, \tau, 1) = O((1 - \lambda)\tau)$, $\tau < 1, \quad \lambda \to 1$. \quad (19)

From (3), (5), (11), (16)–(19) it is not hard to get that

\[ \sigma^2 \geq \begin{cases} C_0 > 0, & \tau > 5/2; \\ C_0(1 - \lambda)^{\tau - 5/2}, & 1 < \tau < 5/2; \end{cases} \quad (20) \]

\[ \sigma^2 \leq \begin{cases} C_0 > 0, & \tau > 3; \\ C_0(1 - \lambda)^{\tau - 3 - \epsilon}, & 1 < \tau \leq 3, \end{cases} \quad (21) \]

\[ |\varphi''(t)| = \begin{cases} O(1), & \tau > 4; \\ O((1 - \lambda)^{\tau - 4 - \epsilon}), & 1 < \tau \leq 4. \end{cases} \quad (22) \]

The next expression is valid for a sufficiently small $t$:

\[ \ln \varphi(t) = t \ln \varphi(t)|_{t=0} + \frac{t^2}{2} \ln \varphi(t)|_{t=0} + \frac{t^3}{6}Q(t), \quad (23) \]

where $|Q(t)| \leq 2 \max_{|u| \leq |t|} |(\ln \varphi(u))''|$. Using (13), (20)–(23) and (A1)–(A5) we get (14).
According to the inversion formula we represent the probability $P\{\zeta_N = k\}$ as the integral

$$P\{\zeta_N = k\} = \frac{1}{\sigma \sqrt{2\pi N}} \int_{-\infty}^{\sigma \sqrt{N}} e^{-i z t} \varphi_N(t) dt,$$

where $z = (k - n)/(\sigma \sqrt{N})$. Since

$$(\sqrt{2\pi})^{-1} e^{-z^2/2} = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i z t - t^2/2} dt,$$

the difference

$$R = 2\pi [\sigma \sqrt{N} P\{\zeta_N = k\} - (2\pi)^{-1/2} e^{-z^2/2}]$$

can be rewritten as the sum of four integrals:

$$R = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_{-\infty}^{A} e^{-i z t} [\varphi_N(t) - e^{-t^2/2}] dt,$$

$$I_2 = \int_{A}^{\infty} e^{-i z t} \varphi_N(t) dt,$$

$$I_3 = \int_{a \sigma \sqrt{N}}^{\infty} e^{-i z t} \varphi_N(t) dt,$$

$$I_4 = - \int_{-\infty}^{-A} e^{-i z t - t^2/2} dt,$$

(25)

the positive constants $A$ and $a$ will be chosen later. Lemma 2 will be proved if we show that by choosing sufficiently large $n, N$ the difference $R$ can be made arbitrarily small. From (14) we get that $I_1 \to 0$. Moreover,

$$|I_4| \leq \int_{A<|t|} e^{-t^2/2} dt,$$

and the integral $I_4$ is as small as desired, provided that $A$ is large enough.

Let us estimate the integral $I_2$. From (23) and (12) we obtain that for sufficiently small $a \ |\varphi_N(t)| \leq e^{-C_1 t^2}$ as $n/N \to 1$, therefore the next estimate is true $|I_2| \leq \int_{A<|t|} e^{-C_1 t^2} dt$, and the integral $I_2$ is small for large enough $A$. From (13), (17) and (20) we obtain the same estimate as $n/N \nrightarrow (1, \tau - 1)/\Sigma(1, \tau), \tau > 4$.

Let $n/N \nrightarrow (1, \tau - 1)/\Sigma(1, \tau), \tau \leq 4$. We divide $I_2$ into integrals $I_2^\prime$ and $I_2^\prime\prime$, where the integration domains are

$$\{t : A < |t| \leq a B(\lambda, \tau) \sigma \sqrt{N}\}$$

and

$$\{t : a B(\lambda, \tau) \sigma \sqrt{N} < |t| \leq a \sigma \sqrt{N}\},$$

where

$$B(\lambda, \tau) = \begin{cases} (1 - \lambda)^{-\tau + 4 + \epsilon}, & 5/2 < \tau \leq 4; \\ (-\ln(1 - \lambda))(1 - \lambda)^{3/2 + \epsilon}, & \tau = 5/2; \\ (1 - \lambda)^{3/2 + \epsilon}, & 1 < \tau < 5/2. \end{cases}$$

From (13) we get that

$$\ln \varphi \left( \frac{t}{\sigma \sqrt{N}} \right) = \frac{itm}{\sigma \sqrt{N}} - \frac{t^2}{2 \sigma^2 N} + \frac{t^3}{6 \sigma^3 N^{3/2}} Q \left( \frac{t}{\sigma \sqrt{N}} \right),$$

where

$$|Q(t/(\sigma \sqrt{N}))| \leq 2 \max_{|u| \leq |t|/(\sigma \sqrt{N})} |\ln'' \varphi(u)|.$$

In the integration domains of the integral $I_2^\prime$ $t/(\sigma \sqrt{N}) \to 0$, then from Lemma 1, (17), (20) and (27) we obtain:

$$\left| \frac{t}{(\sigma^3 N)} Q \left( \frac{t}{\sigma \sqrt{N}} \right) \right| \leq a.$$

It follows that for small enough $a \ |\varphi_N(t)| \leq \exp(-C_{12} t^2)$. Therefore

$$|I_2^\prime| \leq 2 \int_{A<|t|} e^{-C_{12} t^2} dt,$$

and the integral $I_2^\prime$ is as small as desired, provided that $A$ is large enough. To estimate the integral $I_2^\prime\prime$ we expand the function $\Sigma(\lambda z, \tau)$, where $z = e^{it/(\sigma \sqrt{N})}$ in the Taylor series near the point $z = 1$. Then

$$\varphi(t) = 1 - (1 + o(1)) \Sigma(\lambda, \tau - 1) \Sigma^{-1}(\lambda, \tau) \times (1 - \cos(t/(\sigma \sqrt{N})) - i \sin(t/(\sigma \sqrt{N}))).$$

Therefore

$$|\varphi_N(t)| \leq |\varphi_N(t/(\sigma \sqrt{N}))| \leq$$
\[
\leq \exp\{C_{13}N(1 - \cos(t/\sqrt{N}))\}.
\]

Using (27), the conditions (A1) – (A5) and the inequality
\[1 - \cos(t/(\sigma\sqrt{N})) \geq 1 - (1 - C_{14}t^2/(\sigma^2N))\]
we can show that
\[|t| < a\sigma\sqrt{N},\]
\[
|I_2''| \leq \int_{aB(\lambda, \tau)\sigma\sqrt{N}}^{\infty} e^{-C_{14}t^2/\sigma^2} \, dt \leq \]
\[
C_{15} \frac{\sigma}{aB(\lambda, \tau)\sqrt{N}} e^{-C_{14}a^2B^2(\lambda, \tau)N} \to 0.
\]

Let us consider the integral \(I_3\). For \(\varepsilon \leq |t| \leq \pi\) the inequality
\[
|\varphi(t)| \leq e^{-C_{16}}
\]
is valid. Then under the condition that \(n/N \to 1\) it can be shown that
\[
\varphi(t) = e^{it} \left(1 + \lambda p_2(e^{it} - 1)/p_1\right) + o(\lambda^2).
\]
From this and Lemma 1 we get that for \(\varepsilon \leq |t|/(\sigma\sqrt{N}) \leq \pi\)
\[|\varphi(t/(\sigma\sqrt{N}))| \leq e^{-C_{17}\lambda}.
\]
Therefore using relations (12) and (25) it is not hard to see that
\[|I_3| \leq C_{18}\sqrt{n - N} \exp\{-C_{19}(n - N)\} \to 0.
\]

Let \(n/N \to \Sigma(1, \tau - 1)/\Sigma(1, \tau)\). From the conditions (A1) – (A5), (21) and (28) we get that
\[|I_3| \leq C_{20}\sigma\sqrt{N}e^{-C_{21}n} \to 0.
\]

Thus Lemma 2 is proved.

Let \(\varphi_r(t)\) be the characteristic function of the random variable \((\zeta_N^{(r)} - n)/(\sigma\sqrt{N})\).

**Lemma 3.** Let \(n, N \to \infty\). Then uniformly with respect to \(t\) in any fixed finite interval the next conclusions are true

1. if \(n/N \to 1, (n - N)^3/N^2 \to \infty, NP_{r-1} \to \infty, NP_r \to \gamma, \) where \(\gamma\) is a nonnegative constant, then for \(s = 0, \pm 1\) \(\varphi_{r+s}(t) \to e^{-t^2/2};\)

2. if \(n/N \to \Sigma(1, \tau - 1)/\Sigma(1, \tau)\), \(NP_r \to \gamma, \) where \(\gamma\) is a positive constant, parameters \(\tau, N, n\) are determined by the conditions (A1) – (A5) then \(\varphi_r(t) \to e^{-t^2/2}.\)

**Proof.** From (7) and (10) it is easy to see that
\[
\varphi_r(t) = \exp\left\{-i\frac{tn}{\sigma\sqrt{N}}\right\}(1 - P_r)^{-N}\varphi^N\left(\frac{t}{\sigma\sqrt{N}}\right) \times \\
\left(1 - (1 + o(1)) \sum_{k=r+1}^{\infty} p_k(\lambda) \exp\left\{itk/\sigma\sqrt{N}\right\}\right)^N.
\]

It is not hard to get that
\[
\sum_{k=r+1}^{\infty} p_k(\lambda) \exp\left\{tk/\sigma\sqrt{N}\right\} = P_r + R(t),
\]
where \(R(t) \leq |t/\sigma\sqrt{N}| \sum_{k=r+1}^{\infty} p_k(\lambda)k.\)

Let \(n/N \to 1\). It is clear that
\[
NP_r = N \sum_{i \geq 0} p_{r+i+1}(\lambda) = \\
N \left(\sum_{i=0}^{M} p_{r+i+1}(\lambda) + \sum_{i \geq M+1} p_{r+i+1}(\lambda)\right),
\]

the positive constant \(M\) will be chosen later. For the fixed integer \(r\) we get from Lemma 1, (2)–(4) and (15) that
\[
\sum_{i=0}^{M} p_{r+i+1}(\lambda) = \frac{N\lambda^{r+1}h(r + 1)}{\Sigma(\lambda, \tau)(r + 1)^r} (1 + o(1))
\]
and for large enough \(N\)
\[
\sum_{i \geq M+1} p_{r+i+1}(\lambda) = O \left(\sum_{i \geq 0} p_{r+i+1}(\lambda)\right).
\]

Therefore
\[
NP_r = \frac{N\lambda^{r+1}h(r + 1)}{\Sigma(\lambda, \tau)(r + 1)^r} (1 + o(1)).
\]

Using \(NP_r \to \gamma\) we obtain that for fixed integer \(r\)
\[\frac{(\sigma\sqrt{N})^{-1}}{\lambda} \sum_{k \geq r} kp_k(\lambda) = o\left(N^{-1}\right).
\]

As \(r \to \infty\) we can deduce from Lemma 1, (2)–(4) and (15) the relation (33) is valid.

From (2)–(4), (15) and the relation \(NP_r \to \gamma\) we can get that as \(r \to \infty\)
\[
\frac{(\sigma\sqrt{N})^{-1}}{\lambda} \sum_{k \geq r} kp_k(\lambda) \leq C_{22}t(r + 1)p_r(\lambda)/(\sigma\sqrt{N}).
\]
Since \( NP_{r-1} \to \infty \) it is not hard to show that
\[ r = o(\sqrt{n-N}). \]
From this, (31), (35) it follows that
\[ (\sigma \sqrt{N})^{-1} \sum_{k>r} kp_k(\lambda) = o \left( N^{-1} \right). \]
Therefore, for \( n/N \to 1 \) the relation \( \varphi_r(t) \to e^{-t^2/2} \) holds.
For \( s = 1 \) we get that \( NP_{r+1} \to 0 \). Therefore in this case the assertion of Lemma 3 follows from (30) by substituting \( r \) with \( r+1 \).

Let \( s = -1 \). By analogy with the estimate (34) we can obtain that as \( r \to \infty \)
\[ (\sigma \sqrt{N})^{-1} \sum_{k>r} p_k(\lambda)k \leq C_{23} trp_r(\lambda)/(\sigma \sqrt{N}). \]
Using (15) and the condition \( (n-N)^3/N^2 \to \infty \) the relation (34) follows from this and (2)–(4).

By analogy with this estimate for fixed integer \( r \) we can get that
\[ \frac{1}{\sigma \sqrt{N}} \sum_{k>r} p_k(\lambda)k = \]
\[ = \frac{trp_r(\lambda)}{\sigma \sqrt{N}} \sum_{k>r} \lambda^{k-r} \left( \frac{r}{k} \right)^{r-1} \frac{h(k)}{h(r)} \leq \]
\[ \leq C_{24} \frac{trp_r(\lambda)}{\sigma \sqrt{N}} = o \left( \frac{1}{N} \right). \]
Therefore, as \( n/N \to 1 \), the relation \( \varphi_{r-1}(t) \to e^{-t^2/2} \) holds.
Let \( n/N \not\sim \Sigma(1,\tau-1)/\Sigma(1,\tau) \). Using (2)–(4), the properties of the slowly varying function (15), Lemma 1 and (32) we can deduce that
\[ \sum_{i=0}^{M} p_{r+i+1}(\lambda) = p_{r+1}(\lambda) \sum_{i=0}^{M} \lambda^i (1 + o(1)). \quad (36) \]
From (15) it is not hard to get
\[ \sum_{i \geq M+1} p_{r+i+1}(\lambda) = o \left( \sum_{i=0}^{M} p_{r+i+1}(\lambda) \right). \quad (37) \]
From the condition \( NP_r \to \gamma \) it follows that \( r \to \infty \). Then from (32), (36), (37) we get that
\[ NP_r = \frac{N \lambda^{r+1} h(r+1)}{\Sigma(\lambda,\tau)(r+1)^{(1-\lambda)(1+o(1))}}. \quad (38) \]
Using Lemma 1, (15), (38) and the condition \( NP_r \to \gamma \) it is not hard to see that
\[ \frac{t}{\sigma \sqrt{N}} \sum_{k \geq r+1} p_k(\lambda)k \leq C_{25} \frac{t(r+1) p_{r+1}(\lambda)}{\sigma \sqrt{N}(1-\lambda)} \leq \]
\[ \leq C_{26} \frac{t(r+1)}{\sigma N^{3/2}}. \quad (39) \]
From (38) and the condition \( NP_r \to \gamma \) it is easy to see that
\[ \frac{N \lambda^{r+1} h(r+1)}{\Sigma(\lambda,\tau)(r+1)^{(1-\lambda)}} \to \gamma > 0. \quad (40) \]
Using the conditions (A1)–(A5), (20) and (40) we can get that \( (r+1)/(\sigma \sqrt{N}) = o(1) \). From this and (39) we can obtain that
\[ t(\sigma \sqrt{N})^{-1} \sum_{k \geq r+1} p_k(\lambda)k = o \left( N^{-1} \right). \]

Then the assertion of Lemma 3 follows from (30) and (31).

\section*{Lemma 4}

Let \( n, N \to \infty \) and one of the following conditions be fulfilled

1. \( n/N \to 1, (n-N)^3/N^2 \to \infty, NP_{r-1} \to \infty, NP_r \to \gamma, \) where \( \gamma \) is a nonnegative constant;

2. \( n/N \not\sim \Sigma(1,\tau-1)/\Sigma(1,\tau), NP_r \to \gamma, \) where \( \gamma \) is a positive constant and parameters \( \tau, N, n \) are determined by one of the conditions (A1)–(A5).

Then
\[ \mathbf{P}\{\zeta_N^{(r)} = k\} = \frac{1 + o(1)}{\sigma \sqrt{2\pi N}} \exp \left\{ - \frac{(k-n)^2}{2\sigma^2 N} \right\} \]
uniformly with respect to integers \( k \) such that \( (k-n)/\sigma \sqrt{N} \) lies in any fixed finite interval.

\textbf{Proof.} We follow the scheme of proving Lemma 2. We represent the probability \( \mathbf{P}\{\zeta_N^{(r)} = k\} \) as the integral
\[ \mathbf{P}\{\zeta_N^{(r)} = k\} = \frac{1}{2\pi \sigma \sqrt{N}} \int_{-\pi \sigma \sqrt{N}}^{\pi \sigma \sqrt{N}} e^{-izt} \varphi_r(t) dt, \]
where \( z = (k-n)/(\sigma \sqrt{N}) \) and \( \varphi_r(t) \) is the characteristic function of the random variable \( \zeta_N^{(r)} - n/\sigma \sqrt{N} \). Using (24) the difference
\[ R = 2\pi [\sigma \sqrt{N} \mathbf{P}\{\zeta_N^{(r)} = k\} - (2\pi)^{-1/2} e^{-t^2/2}] \]
can be rewritten as the sum of four integrals:
\[ R = I_1^{(r)} + I_2^{(r)} + I_3^{(r)} + I_4, \] where \( I_4 \) is given by (25) and \( I_1^{(r)} - I_3^{(r)} \) are constructed similarly to \( I_1 - I_3 \) by substituting \( \varphi_r(t) \) instead of \( \varphi_N(t) \) in (25).
From Lemma 3 it follows that \( I_1^{(r)} \to 0 \). From, (12), (20)–(23) and (30) we get that
\[
|\varphi_r(t)| \leq (1-P_r)^{-N} (\exp \{ C_{27}t^2/N \} + C_{28}/N)^N.
\]
Therefore, \( |I_2^{(r)}| \leq C_{29} \int_0^\infty e^{-C_{30}t^2} dt \). It is clear that \( I_2^{(r)} \) can be made arbitrarily small by choosing a sufficiently large \( A \).

It is easy to show that we can estimate the integral \( I_3^{(r)} \) by analogy with \( I_3 \) in Lemma 2. For \( I_4 \) we can use the estimation (26). This completes the proof of Lemma 4.

It is not hard to see that the conclusion of Lemma 4 is true when replacing \( r \) with \( I \)

choosing a sufficiently large \( I \)

Therefore, \( I_2^{(r)} \) can be made arbitrarily small by choosing a sufficiently large \( A \).

Using Poisson approximation of the binomial distribution as \( p_r(\lambda) \to 0 \), Lemmas 2, 6 and relations (9), (41) we can obtain the assertion of Theorem 4.

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**REFERENCES**


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