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# ON THE CLUSTERING COEFFICIENT OF CONFIGURATION GRAPHS

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Configuration graphs with $N$ vertices are considered. The degrees of the vertices are independent identically distributed random variables following the power-law distribution with a positive parameter $\tau$, where $\tau = \tau(N)$ varies as $0 < c_1 \leqslant \tau \leqslant c_2 < \infty$ and can take both fixed values and non-fixed values. Theorems describing the limit behaviour of the clustering coefficient for such graphs as $N \to \infty$ are formulated.

Keywords: configuration graph; clustering coefficient; limit theorems

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## И. А. Чеплюкова. О КОЭФФИЦИЕНТЕ КЛАСТЕРИЗАЦИИ КОНФИГУРАЦИОННЫХ ГРАФОВ

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Рассматриваются конфигурационные графы с $N$ вершинами. Степени вершин графа являются независимыми одинаково распределенными случайными величинами, распределение которых является степенным распределением с положительным параметром $\tau$, где $\tau = \tau(N)$ изменяется в диапазоне $0 < c_1 \leqslant \tau \leqslant c_2 < \infty$ и может принимать не только фиксированные значения. Получены теоремы, описывающие предельное поведение коэффициента кластеризации для таких графов с числом вершин $N$, стремящимся к бесконечности.

Ключевые слова: конфигурационный граф; кластерный коэффициент; предельные теоремы

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Финансирование. Финансовое обеспечение исследований осуществлялось из средств федерального бюджета на выполнение государственного задания КарНЦ РАН (Институт прикладных математических исследований КарНЦ РАН).

## Introduction

Random graphs have been widely used to model complex communication networks such as mobile networks, Internet, transport, social networks, etc. (e. g. [5, 7]). One of the most commonly used classes of random graphs is the configuration model. This model was first introduced in [3].

This paper deals with the configuration graph proposed in [11] with the vertex degrees being independent identically distributed random variables following the distribution:

$$\mathbf{P}\{\xi \geqslant k\} = \frac{h(k)}{k^{\tau}},\ \tau \in (1,2),\ k = 1, 2, \ldots, \quad (1)$$

where the random variable $\xi$ is equal to the degree of any vertex and $h(x)$ is a slowly varying function, i. e. $h(ax)/h(x) \to 1$ for every $a > 0$ and $x \to \infty$.

The construction of the configuration model can be described as follows. Random variables equal to vertex degrees are drawn independently from the distribution (1). The degree of each vertex in the configuration graph is equal to the number of its incident semiedges. All semiedges are numbered in an arbitrary order. Obviously, the sum of vertex degrees has to be even. Otherwise, an auxiliary vertex with degree one is added. The graph is constructed by joining all semiedges pairwise equiprobably to form edges. Because pairing is done without restrictions, multiple edges and loops can appear.

There are many works (e. g. [4, 11]) where results describing the limit behaviour of configuration graphs were obtained. Attention in the studies of configuration graphs has been given not only to the properties of the degree structure, but also to other numerical characteristics (see [8]). One of such graph characteristics is the clustering coefficient.

For the graph $G$ the clustering coefficient $C_G$ can be defined as follows (see [8])

$$C_G = \frac{3 \times \textit{number of graph triangles}}{\textit{number of connected triples of vertices}},$$

where a "connected triple" means a single vertex connected by edges to two others. In effect, $C_G$ measures the fraction of triples that have their third edge filled in to complete a triangle. Here we define this notion for random graphs. We will use the terminology adopted in [7]. Consider the graph $G = G(V, E)$ with $N$ vertices where $V$ is the set of its vertices and $E$ is the set of its edges. We say that the distinct vertices, $(i, j, k)$ form an occupied triangle when the edges $ij$, $jk$ and $ki$ are all occupied. Note that $(i, j, k)$ is the same triangle as $(i, k, j)$ and as any other permutation. Following [7] (see equation (4.7.1)), we define the clustering coefficient of a random graph $G$ to be

$$C_G = \frac{\mathbf{E}(\Delta_G)}{\mathbf{E}(W_G)},$$

where

$$\Delta_G = \sum_{i,j,k \in V} I\{ij, jk, ki \text{ occupied}\},$$

$$W_G = \sum_{i,j,k \in V} I\{ij, ik \text{ occupied,}\},$$

$I\{A\}$ is the indicator of an event $A$. Thus, $\Delta_G$ is six times the number of triangles in $G$, and $W_G$ is two times the number of adjacent edges in $G$, and $C_G$ is the ratio of the number of expected triangles to the expected number of adjacent edges. Note that the number of triangles formed by three vertices can be more than one if the edges between the vertices are multiples.

It is proved in [8] that for this configuration graph the clustering coefficient $C_G$ is

$$C_G = \frac{\left(\sum\limits_{k=1}^{\infty} k q_k\right)^2}{Nm}, \quad (2)$$

where $N$ is equal to the number of vertices of the graph,

$$p_k = \mathbf{P}\{\xi = k\},$$
$$q_k = (k+1)p_{k+1}/m, \qquad k = 0, 1, 2, \ldots,$$
$$m = \sum_{k=1}^{\infty} k p_k.$$

It was noted in [11] that the function $h(k)$ does not affect the main asymptotic properties of the configuration graph as $N \to \infty$. So, the authors of [11] suggest to use the simplest case $h(k) = 1$. In this case

$$p_k = \mathbf{P}\{\xi = k\} = k^{-\tau} - (k+1)^{-\tau}, \quad k = 1, 2, \ldots. \quad (3)$$

It follows from (3) that as $k \to \infty$

$$p_k \sim \frac{\tau}{k^{\tau+1}}. \quad (4)$$

In [9] Pavlov considered a configuration graph in which the distribution of vertex degrees must meet only the following condition as $k \to \infty$:

$$\mathbf{P}\{\xi = k\} \sim \frac{d}{k^g \ln^h k}, \qquad (5)$$

where $d > 0$, $g > 1$, $h \geqslant 0$. Obviously, by virtue of (4), the distribution (3) satisfies the condition (5) as $d = \tau$, $g = \tau + 1$, $h = 0$. In [9], the limit theorems for the clustering coefficient $C_G$ of such graphs with fixed parameters $g$ and $h$ were formulated. However, it was noted that the vertex degree distribution can change as the network size grows (e. g. [2]). In [10], conditional configuration graphs are considered where the degrees of the vertices are independent identically distributed random variables following the power-law distribution and the parameter of this distribution is a random variable uniformly distributed on the interval $[a, b]$, $0 < a < b < \infty$. The limit distributions of the number of vertices with a given degree were obtained in [10].

Here we consider a configuration graph in which vertex degrees have the distribution (3), where the distribution parameter $\tau$ can vary in the interval $0 < c_1 \leqslant \tau \leqslant c_2 < \infty$ and is not necessarily fixed. The aim is to study the limit behavior of the clustering coefficient $C_G$ for such configuration graphs (Theorem 1 and Theorem 2).

Theorem 1 describes the limit behavior of the clustering coefficient $C_G$ for the case of the parameter $\tau > 2$ as $N \to \infty$. In this case, the variance $\mathbf{D}\xi$ of a random variable equal to the vertex degree is finite. Theorem 2 describes the limit behavior of $C_G$ as $0 < c_1 \leqslant \tau \leqslant 2$. It is not hard to see that in this case the variance $\mathbf{D}\xi$ is infinite. Here, we will use the notion "asymptotically almost sure" (a.a.s.), which means the following. Let $A_N$ be an event of a random graph with $N$ vertices having a certain property. We say that $A$ happens a.a.s., if

$$\lim_{N\to\infty} \mathbf{P}\{A_N\} = 1.$$

Let the random variables $\xi_1, \xi_2, \ldots, \xi_N$ be equal to the degrees of vertices with numbers $1, 2, \ldots, N$ and

$$\xi_{(N)} = \max\{\xi_1, \ldots, \xi_N\}.$$

Observe that the maximum vertex degree of our graphs is proportional to $N^{1/\tau}$ a.a.s. (see Lemma below). For the case of $0 < c_1 \leqslant \tau \leqslant 2$ we restrict ourselves to considering random graphs that satisfy this condition. Namely, in Theorem 2 we will consider conditional configuration graphs under the condition

$$\xi_{(N)} \leqslant uN^{1/\tau}, \qquad 0 < u < \infty.$$

Theorem 2 shows that the asymptotically clustering coefficient of a random graph depends on the maximum vertex degree.

The article has the following structure. Section 2 formulates the main results (Theorems 1 and 2). In section 3 these theorems are proved.

## Main results

Consider a configuration graph in which the vertex degrees $\xi_1, \xi_2, \ldots, \xi_N$ are independent identically distributed random variables with the distribution (3). If the parameter $\tau > 2$, then the following assertion is true.

**Theorem 1.** *Let $N \to \infty$. Then the relations*

$$C_G = \begin{cases} 4\left(\zeta(\tau-1)-\zeta(\tau)\right)^2/(N\zeta^3(\tau)), \\ \qquad\qquad \textit{if} \quad \tau = \tau(N) \geqslant c_3 > 2; \\ 4(\tau-2)^{-2}/(N\zeta^3(\tau))(1+o(1)), \\ \qquad\qquad \textit{if} \quad \tau = \tau(N) \searrow 2, \end{cases}$$

*hold, where $\zeta(x)$ is the value of the Riemann zeta function at the point $x$.*

**Corollary 1.** *It follows from Theorem 1 that*

1. *if $\tau \geqslant c_3 > 2$ or $\tau = \tau(N) \searrow 2$, $(\tau-2)^2 N \to \infty$, then $C_G \to 0$;*
2. *if $\tau = \tau(N) \searrow 2$, $(\tau-2)^2 N \to const \neq 0$, then $C_G$ tends to some positive constant;*
3. *if $\tau = \tau(N) \searrow 2$, $(\tau-2)^2 N \to 0$, then $C_G \to \infty$.*

Let $0 < c_1 < \tau \leqslant 2$. Consider conditional configuration graphs under the condition that

$$\xi_{(N)} \leqslant uN^{1/\tau}, 0 < u < \infty.$$

For such conditional graphs the assertions below are true.

**Theorem 2.** *Let $N \to \infty$. Then*

1. *if $\tau = 2$, then*

$$C_G = \frac{e^{-u^{-2}}}{\zeta^3(2)} \frac{\ln^2 N}{N} \qquad \textit{a.a.s.;}$$

2. *if $\tau \nearrow 2$, then*

$$C_G = \frac{4\exp\{-u^{-2}\}(N^{1-\tau/2} - 2/\tau)^2}{\zeta^3(2)(2-\tau)^2 N}$$

*a.a.s.;*

3. *if* $1 < c_4 \leqslant \tau \leqslant c_5 < 2$, *then*

$$C_G = \frac{\exp\{-u^{-\tau}\}}{\zeta^3(\tau)} \left(\frac{u^{2-\tau}\tau}{2-\tau}\right)^2 N^{(4-3\tau)/\tau}$$

*a.a.s.;*

4. *if* $\tau \searrow 1$, *then*

$$C_G = \exp\{-u^{-1}\}(\tau-1)^3 u^2 N^{(4-3\tau)/\tau}$$

*a.a.s.;*

5. *if* $\tau = 1$, *then*

$$C_G = \exp\{-u^{-1}\}\frac{u^2 N}{\ln^3 N} \qquad a.a.s.;$$

6. *if* $0 < c_1 \leqslant \tau < 1$, *then*

$$C_G = \frac{e^{-u^{-\tau}}(1-\tau)^3 u^{1+\tau} N^{1/\tau}}{\tau(2-\tau)^2}$$

*a.a.s.*

**Corollary 2.** *It follows from Theorem 2 that the following assertions are true a.a.s.*

1. *if* $(\tau - 4/3)\ln N \to +\infty$, *then* $C_G = 0$*;*
2. *if* $|(\tau - 4/3)\ln N| \leqslant c_6 < \infty$, *then* $C_G$ *is equal to the constant and this constant depends on* $u$*;*
3. *if* $(\tau - 4/3)\ln N \to -\infty$, *then* $C_G = \infty$.

## Proofs of the main results

First, we will prove Theorem 1. Let $\tau \geqslant c_3 > 2$. In this case, $m = \zeta(\tau)$ and it follows from (2) that

$$C_G = \frac{\left(\sum\limits_{k=1}^{\infty} kq_k\right)^2}{\zeta(\tau)N} = \frac{1}{\zeta^3(\tau)N}$$

$$\times\left(\sum_{k=1}^{\infty}(k^2+k)\left(\frac{1}{(k+1)^\tau} - \frac{1}{(k+2)^\tau}\right)\right)^2$$

$$= \frac{1}{\zeta^3(\tau)N}\left(\sum_{k=2}^{\infty}(2k-2)\frac{1}{k^\tau}\right)^2. \qquad (6)$$

This yields the assertion of Theorem 1 for the case $\tau \geqslant c_3 > 2$.

Let $\tau = \tau(N) \searrow 2$. We will use the known expansion of the zeta-function about point 1 (see [1]):

$$\zeta(1+y) = y^{-1} + c + O(y), \quad y > 0,\ y \to 0, \quad (7)$$

where $c$ is the Euler–Mascheroni constant. It follows from here and (6) that

$$C_G = \frac{4}{\zeta^3(\tau)N}(\zeta(\tau-1) - \zeta(\tau))^2$$

$$= \frac{4\zeta^2(\tau-1)}{\zeta^3(\tau)N}(1+o(1))$$

$$= \frac{4\left((\tau-2)^{-1} + c + O(\tau-2)\right)^2}{\zeta^3(\tau)N}(1+o(1))$$

$$= \frac{4(\tau-2)^{-2}}{\zeta^3(\tau)N}(1+o(1)).$$

Therefore, Theorem 1 is fully proven.

Let us now prove Theorem 2. Here we consider the conditional configuration graphs under the condition that

$$\xi_{(N)} \leqslant uN^{1/\tau}, 0 < u < \infty.$$

Clearly, for a detailed study of the clustering coefficient $C_G$ it is desirable to know the limit distribution of the maximum degree of graph vertices. The next Lemma follows from (3) and the known classical result [6].

**Lemma.** *Let* $N \to \infty$. *Then*

$$\mathbf{P}\{\xi_{(N)} \leqslant xN^{1/\tau}\} = e^{-x^{-\tau}} + o(1), \quad x > 0$$

*holds.*

According to Lemma for the maximal degree $\xi_{(N)}$, the inequation $\xi_{(N)} > N^{1/(\tau+\varepsilon)}$, $\varepsilon > 0$ holds a.a.s. Moreover, it follows from Lemma that

$$\mathbf{P}\left\{N^{1/(\tau+\varepsilon)} \leqslant \xi_{(N)} \leqslant \frac{1}{\delta}N^{1/\tau}\right\}$$

can be made arbitrarily close to 1 by choosing sufficiently small positive $\varepsilon$ and $\delta$. We can then say that Lemma predicates that the maximum vertex degree $\xi_{(N)}$ is proportional to $N^{1/\tau}$ a.a.s. It thus seems reasonable to consider the set of graphs in which $\xi_{(N)} \leqslant uN^{1/\tau}$, where

$$N^{-\frac{\varepsilon}{\tau(\tau+\varepsilon)}} < u < 1/\delta.$$

So, by virtue of Lemma, the condition for the maximal degree of our graphs in Theorem 2 is natural.

It follows from (2) that the clustering coefficient $C_G$ for such conditional configuration graphs can be obtained from the following relation:

$$C_G = \frac{\left(\sum\limits_{k=1}^{\infty} kq'_k\right)^2}{Nm'}, \qquad (8)$$

where

$$p'_k = \mathbf{P}\{\xi = k | \xi_{(N)} \leqslant uN^{1/\tau}\},$$

$$q'_k = (k+1)p'_{k+1}/m', \quad k = 0,1,2,\dots,$$

$$m' = \sum_{k=1}^{\infty} k p'_k.$$

Then, combining (3) and (8) we get

$$C_G = \frac{1}{N}\mathbf{P}\left\{\xi_{(N)} \leqslant uN^{1/\tau}\right\} \times \frac{\left(\sum_{k=1}^{[uN^{1/\tau}]-1} k(k+1)p_{k+1}\right)^2}{\left(\sum_{k=1}^{[uN^{1/\tau}]} kp_k\right)^3}, \tag{9}$$

where $[x]$ is the integer part of the number $x$.

Using Lemma, it is not difficult to see that

$$\mathbf{P}\left\{\xi_{(N)} \leqslant uN^{1/\tau}\right\} = e^{-u^{-\tau}}(1+o(1)). \tag{10}$$

From (9) and (10) it follows that

$$C_G = \frac{e^{-u^{-\tau}}}{N} \times \frac{\left(\sum_{k=1}^{[uN^{1/\tau}]-1} k(k+1)p_{k+1}\right)^2}{\left(\sum_{k=1}^{[uN^{1/\tau}]} kp_k\right)^3}(1+o(1)). \tag{11}$$

Using (3), we can show that

$$\sum_{k=1}^{[uN^{1/\tau}]} kp_k = \sum_{k=1}^{[uN^{1/\tau}]} \frac{1}{k^\tau} - \frac{[uN^{1/\tau}]}{\left([uN^{1/\tau}]+1\right)^\tau} \tag{12}$$

and

$$\sum_{k=1}^{[uN^{1/\tau}]-1} k(k+1)p_{k+1} = \sum_{k=2}^{[uN^{1/\tau}]} (2k-2)\frac{1}{k^\tau} - \frac{\left([uN^{1/\tau}]-1\right)^2}{\left([uN^{1/\tau}]+1\right)^\tau} - \frac{[uN^{1/\tau}]-1}{\left([uN^{1/\tau}]+1\right)^\tau}. \tag{13}$$

Let $\tau = 2$. From (13) it follows that

$$\sum_{k=1}^{[uN^{1/2}]-1} k(k+1)p_{k+1} = 2\sum_{k=2}^{[uN^{1/2}]} \frac{1}{k} - 2\sum_{k=2}^{[uN^{1/2}]} \frac{1}{k^2} - \frac{\left([uN^{1/2}]-1\right)^2}{\left([uN^{1/2}]+1\right)^2} - \frac{[uN^{1/2}]-1}{\left([uN^{1/2}]+1\right)^2}. \tag{14}$$

Using a well-known formula

$$\sum_{k=1}^{[N]} \frac{1}{k} = \ln N + c + \varepsilon(N) \tag{15}$$

where $\varepsilon(N) \sim 1/(2N)$, we get

$$\sum_{k=2}^{[uN^{1/2}]} \frac{1}{k} = \frac{1}{2}\ln N(1+o(1) \tag{16}$$

then from (11)–(16) it follows that for $\tau = 2$

$$C_G = \frac{\exp\{-u^{-2}\}\ln^2 N}{\zeta^3(2)N}(1+o(1)). \tag{17}$$

Let $\tau \nearrow 2$. Using (13), we get

$$\sum_{k=1}^{[uN^{1/\tau}]-1} k(k+1)p_{k+1} = 2\sum_{k=1}^{[uN^{1/\tau}]} \frac{1}{k^{\tau-1}}(1+o(1)) - \left[uN^{1/\tau}\right]^{2-\tau}(1+o(1)). \tag{18}$$

It is easy to see that

$$\sum_{k=2}^{[uN^{1/\tau}]} \frac{1}{k^{\tau-1}} = (1+o(1))N^{(2-\tau)/\tau} \int\limits_{2/N^{1/\tau}}^{u} y^{-\tau+1}dy = (1+o(1))\frac{N^{(2-\tau)/\tau}}{2-\tau}\left(u^{2-\tau} - \frac{2^{2-\tau}}{N^{(2-\tau)/\tau}}\right) = \frac{1}{2-\tau}\exp\left\{\frac{2-\tau}{\tau}\ln N\right\} \times \left(1-\exp\left\{-\frac{2-\tau}{\tau}\ln N\right\}\right)(1+o(1)). \tag{19}$$

This relation together with (11), (12) and (18) yields

$$C_G = \frac{\exp\{-u^{-2}\}}{\zeta^3(2)N} \times \left(\frac{2(\exp\{\frac{2-\tau}{\tau}\ln N\}-1)(1+o(1))}{2-\tau} - [uN^{1/\tau}]^{2-\tau}(1+o(1))\right)^2$$

$$= \frac{\exp\{-u^{-2}\}(1+o(1))}{\zeta^3(2)N}\left(2\frac{\exp\{\frac{2-\tau}{\tau}\ln N\}-1}{2-\tau}\right.$$

$$-\exp\left\{\frac{2-\tau}{\tau}\ln N\right\}\Bigg)^2$$

$$=\frac{\exp\{-u^{-2}\}\left(-2+\tau\exp\left\{\frac{2-\tau}{\tau}\ln N\right\}\right)^2}{(2-\tau)^2\zeta^3(2)N}(1+o(1))$$

$$=\frac{4e^{-u^{-2}}(N^{1-\tau/2}-2/\tau)^2}{(2-\tau)^2\zeta^3(2)N}(1+o(1)). \quad (20)$$

Let $1<c_4\leqslant\tau\leqslant c_5<2$. Using relations (13), (18), and the first equality of (19) we get that

$$\sum_{k=1}^{[uN^{1/\tau}]-1} k(k+1)p_{k+1}$$

$$=\frac{\tau}{2-\tau}u^{2-\tau}N^{(2-\tau)/\tau}(1+o(1)).$$

From here and (11) it is easy to see that

$$C_G=\left(e^{-u^{-\tau}}\zeta^3(\tau)N\right)^{-1}\Bigg(2N^{(2-\tau)/\tau}\frac{u^{2-\tau}}{(2-\tau)}$$

$$-(uN^{1/\tau})^{2-\tau}\Bigg)^2(1+o(1)) \quad (21)$$

$$=\left(e^{-u^{-\tau}}\zeta^3(\tau)\right)^{-1}N^{(4-3\tau)/\tau}\left(\frac{u^{2-\tau}\tau}{2-\tau}\right)^2(1+o(1)).$$

Let $\tau\searrow 1$. In the same way as in the previous case, using the known expansion of the zeta-function about point 1 (7) we get that

$$C_G=\left(e^{-u^{-\tau}}\zeta^3(\tau)N\right)^{-1}$$

$$\times\left(2N^{(2-\tau)/\tau}\frac{u^{2-\tau}}{(2-\tau)}-(uN^{1/\tau})^{2-\tau}\right)^2(1+o(1))$$

$$=\frac{e^{-u^{-1}}(\tau-1)^3}{N}\left(u^{2-\tau}N^{(2-\tau)/\tau}\right)^2(1+o(1))$$

$$=e^{-u^{-1}}(\tau-1)^3u^2N^{(4-3\tau)/\tau}(1+o(1)). \quad (22)$$

Let $\tau=1$. From (12) and (15) we get that

$$\sum_{k=1}^{[uN]}kp_k=\sum_{k=1}^{[uN]}\frac{1}{k}-1+o(1)$$

$$=(1+o(1))\ln N. \quad (23)$$

Using (3) and (15) we can find that

$$\sum_{k=1}^{[uN]-1}k(k+1)p_{k+1}=\sum_{k=1}^{[uN]-1}\left(1-\frac{2}{k+2}\right)$$

$$=uN(1+o(1)).$$

From this and (11), (23) it follows that

$$C_G=e^{-1/u}\frac{u^2N}{\ln^3 N}(1+o(1)). \quad (24)$$

Let us consider the last case where $0<c_1\leqslant\tau<1$. It is not hard to see that in this case

$$\sum_{k=1}^{[uN^{1/\tau}]}\frac{1}{k^\tau}=N^{-(\tau-1)/\tau}\int\limits_{1/N^{1/\tau}}^{u}\frac{1}{y^\tau}dy(1+o(1))$$

$$=\frac{u^{1-\tau}N^{(1-\tau)/\tau}}{1-\tau}(1+o(1)). \quad (25)$$

From this and (12) we get that

$$\sum_{k=1}^{[uN^{1/\tau}]}kp_k=\sum_{k=1}^{[uN^{1/\tau}]}\frac{1}{k^\tau}-\frac{[uN^{1/\tau}]}{\left([uN^{1/\tau}]+1\right)^\tau}$$

$$=u^{1-\tau}N^{(1-\tau)/\tau}\frac{\tau}{1-\tau}(1+o(1)). \quad (26)$$

Using (13), the second equality of (19), and (25) we get that

$$\sum_{k=1}^{[uN^{1/\tau}]}k(k+1)p_{k+1}=2\sum_{k=2}^{[uN^{1/\tau}]}\frac{1}{k^{\tau-1}}-2\sum_{k=2}^{[uN^{1/\tau}]}\frac{1}{k^\tau}$$

$$-\left(uN^{1/\tau}\right)^{2-\tau}(1+o(1))$$

$$=u^{2-\tau}N^{(2-\tau)/\tau}\frac{\tau}{2-\tau}(1+o(1)).$$

From (11), (26), and the previous relation we get

$$C_G=\frac{\exp\{-u^{-\tau}\}(1-\tau)^3u^{1+\tau}}{\tau(2-\tau)^2}N^{1/\tau}(1+o(1)).$$

Taking into account the above considerations on the maximum vertex degree, the assertion of Theorem 2 follows from the reasoning here and (17), (20)–(22), (24).

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